1. (Q19, p94) $f: I\times J \to \mathbb{R}$ under suitable conditions.

$$\frac{d}{dt}\int_I f(x,t)\,dx = \int_I \left(\frac{\partial}{\partial t} f(x,t)\right) dx \quad \text{, i.e.}$$

$\forall\, t_0 \in J$

$$\lim_{t\to t_0} \int_I \frac{f(x,t)-f(x,t_0)}{t-t_0}\,dx = \int_I \left(\lim_{t\to t_0} \frac{f(x,t)-f(x,t_0)}{t-t_0}\right) dx$$

By seq. criterion, $\forall\, t_0 \in J$, $\forall\, t_n \to t_0$

$$\lim_{n\to\infty} \int_I \frac{f(x,t_n)-f(x,t_0)}{t_n - t_0}\,dx = \int_I \left(\lim_{n\to\infty} \frac{f(x,t_n)-f(x,t_0)}{t_n-t_0}\right) dx$$

2. (Q14, p93). $f \in L(E)$ with $m(E) \le +\infty$, $\varepsilon > 0$. Then $\exists$ "nice-functions" (simple φ, step ψ, cts g from $\mathbb{R}\to\mathbb{R}$ vanishing outside $E \cap (a,b)$) such that

$$\|\varphi - f\| \left(= \int_E |\varphi - f|\right) < \varepsilon, \ \cdots$$

3. (Q16, p94) Riemann-Lebesgue Th: Let $f \in L(\mathbb{R})$

Then $\int_{-\infty}^{\infty} f(x)\sin nx\,dx$ (Fourier coefficients) $\to 0$

as $n \to \infty$.

Pf. I. $f = \chi_{(a,b)}$

II $f =$ step-function vanishing outside a finite interval

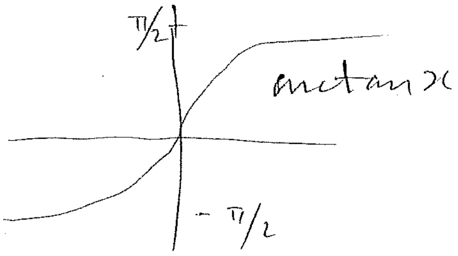
III General $f \in L_1(\mathbb{R})$ (using Q14).

4. Assume 2050+2060 (such that $\uparrow$ of some functions).

(a) $\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}\,dx \overset{?}{=} \int_0^\infty \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n e^{-2x}\,dx$

$$= \int_0^\infty e^x \cdot e^{-2x}\,dx = e^{-x}\Big|_\infty^0 = 1$$

4 (b). $\lim_{n\to\infty}\int_{-\pi/2}^{\pi/2} \sin x \arctan(nx)\, dx$



$\overset{\text{even}}{\underset{\text{funct}}{=}} 2 \lim_{n\to\infty} \int_0^{\pi/2} \sin x \arctan(nx)\, dx$

$\overset{?}{=} 2 \int_0^{\pi/2} \lim_{n\to\infty} \left[\sin x \arctan(nx)\, dx\right] dx$

$= 2\int_0^{\pi/2} \sin x \cdot \frac{\pi}{2}\, dx = \pi$. (Justification BCT)

(c) $\lim_{x\to\infty} \int_1^\infty \frac{\sqrt{x}}{1+nx^3}\, dx \overset{?}{=} \int_1^\infty \left(\lim_{x\to\infty} \frac{\sqrt{x}}{1+nx^3}\right) dx = 0$

Justification: $0 \le f_n(x) \le \frac{\sqrt{x}}{nx^3} = \frac{1}{n}\frac{1}{x^{5/2}}$ (Cauchy-Riemann integrable over $[1,\infty)$)

5. Cauchy-Riemann (improper) integral over $[a,\infty)$. If $f \in \mathcal{R}[a,b]$ $\forall b>a$

$(CR)\int_a^\infty f(x)\,dx \underset{\text{exists}}{\overset{\text{def}}{=}} \lim_{b\to\infty}(R)\int_a^b f(x)\,dx = \lim_{b\to\infty}(\mathcal{L})\int_a^b f(x)\,dx$

$= \lim_{n\to\infty}(\mathcal{L})\int_a^\infty f(x)\chi_{[a,b_n]}(x)\,dx$ $\left(\begin{matrix} b_n \to +\infty \\ b_n \uparrow \end{matrix}\right)$

$\overset{\text{MCT}}{=} (\mathcal{L})\int_a^\infty f(x)\,dx$ $\left(0 \le f(x)\chi_{[a,b_n]} \uparrow_n f(x) \text{ as } f \ge 0\right)$

Similar result for (CR)- integral of f on $[a, \bar{b}]$:

$f \in \mathcal{R}[a,b]$ $\forall\, b \in (a,\bar{b})$ and $\lim_{\substack{b\to\bar{b} \\ a<b<\bar{b}}} (R)\int_a^b f$ exists in $[-\infty,\infty]$

(to be denoted by $(CR)\int_a^{\bar{b}} f$). If $0 \le f$ on $[a,\bar{b}]$ then f is Lebesgue-integrable & the two integrals same.